ON PERIODIC SOLUTIONS OF DYNAMIC, SECOND ORDER, NEARLY PIECEWISE ANALYTIC SYSTEMS

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Necessary and sufficient conditions of existence and stability of periodic solutions of various types are obtained for a particular type of second order, nearly piecewise analytic dynamic systems.

Let us consider the system

$$dx/dt = y, \quad dy/dt = -\psi(x) + \mu f(x, y) \tag{1}$$

and let

$$\psi(x) = \psi_i(x) \quad \text{when } x_{i-1} < x < x_i \qquad f(x, y) = f_i^{(1)}(x, y) \quad \text{when } x_{i-1} < x < x_i,\ y > 0$$

$$f(x, y) = f_i^{(2)}(x, y) \quad \text{when } x_{i-1} < x < x_i,\ y < 0 \qquad (i = \ldots -1, 0, 1, \ldots)$$

Here $\psi_i(x)$ and $f_i^{(j)}(x, y)$ $(j = 1, 2)$ are analytic functions and μ is a small positive parameter. We assume that at the coordinate origin $(x = 0, y = 0)$ the system (1), has the state of equilibrium of the center or "joined center" type.

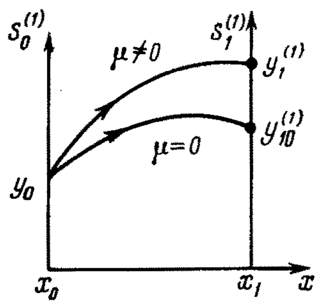

Fig. 1

Let us denote by $S_i^{(1)}$ the lines $x = x_i$ for $y > 0$ and by $S_i^{(2)}$ the lines $x = x$ for $y < 0$ and let us consider phase trajectories of the system (1) when $\mu = 0$ and when $\mu \neq 0$, satisfying in both cases the same initial conditions

$$x = x_0, \quad y = y_0 \text{ when } t = 0 \tag{2}$$

Assuming that the trajectories of (1) intersect the lines $S_k^{(j)}$ at the points $P_{k0}^{(j)}$ $(x_k, y_{k0}^{(j)})$ when $\mu = 0$ and at $P_k^{(j)}$ $(x_k, y_k^{(j)})$ when $\mu \neq 0$, we shall prove that

$$y_k^{(j)} = y_{k0}^{(j)} + \frac{\mu}{y_{k0}^{(j)}} \int\limits_{L_k^{(j)}} f(x, y)\, dx + \mu^2(\ldots) \tag{3}$$

where $L_k^{(j)}$ is the integral curve of (1) passing, at $\mu = 0$, from the point $P_0(x_0, y_0)$ to the point $P_{k0}^{(j)}$ $(x_k, y_{k0}^{(j)})$.

We shall prove first that Formulas (3) hold when the line $S_0^{(1)}$ is transformed into the line $S_1^{(1)}$ (Fig. 1).

Solution of (1) satisfying the initial conditions (2) can be written, when $\mu = 0$, as

$$x = x_1(h_0, t + \varphi_0), \quad y = y_1(h_0, t + \varphi_0) \tag{4}$$

where h_0 and φ_0 are constants.

Considering that the system (1) has the integral

$$H_1(x, y) \equiv {}^1/_2 y^2 + \int \psi_1(x)\, dx = h_0 \qquad (x_0 < x < x_1)$$

when $\mu = 0$, we can write a solution for this system when $\mu \neq 0$ which will satisfy the initial conditions (2), in the following form

$$x = x_1 [\alpha_0(t),\ t + \beta_0(t)] \equiv \xi_1(t), \qquad y = y_1 [\alpha_0(t),\ t + \beta_0(t)] \equiv \eta_1(t) \tag{5}$$

Here $\alpha_0(t)$ and $\beta_0(t)$ represent a solution of

$$\frac{d\alpha_0}{dt} = \mu f_1^{(1)}[\xi_1(t),\ \eta_1(t)] \frac{\partial x_1}{\partial t}, \qquad \frac{d\beta_0}{dt} = -\mu f_1^{(1)}[\xi_1(t),\ \eta_1(t)] \frac{\partial x_1}{\partial h_0} \tag{6}$$

satisfying the initial conditions $\alpha_0(t) = h_0$ and $\beta_0(t) = \varphi_0$ when $t = 0$.

Writing $\alpha_0(t)$ and $\beta_0(t)$ as power series in μ, we obtain

$$\alpha_0(t) = h_0 + \mu\alpha_{01}(t) + \mu^2(\ldots), \qquad \beta_0(t) = \varphi_0 + \mu\beta_{01}(t) + \mu^2(\ldots)$$

where

$$\alpha_{01}(t) = \int_0^t f_1^{(1)} [x_1(h_0,\ t + \varphi_0),\ y_1(h_0,\ t + \varphi_0)] \frac{\partial x_1}{\partial t} dt \tag{7}$$

(explicit expression for $\beta_{01}(t)$ shall not be utilised, since it can be eliminated from the equations).

Let $t_1^{(1)}$ be the least time in which the representative point moving along the trajectory of (1) reaches the line $S_1^{(1)}$ at the point $P_1^{(1)}$ $(x_1,\ y_1^{(1)})$.

Putting $t = t_1^{(1)}$ in (5) and expanding the resulting relation into a power series in μ, we obtain

$$t_1^{(1)} = t_{10}^{(1)} + \mu t_{11}^{(1)} + \mu^2(\ldots)$$

$$x_1 = x_1 + \mu \left[y_{10}^{(1)} t_{11}^{(1)} + \frac{\partial x_1}{\partial h_0} \alpha_{01}(t_{10}^{(1)}) + y_{10}^{(1)} \beta_{01}(t_{10}^{(1)}) \right] + \mu^2(\ldots)$$

$$y_1^{(1)} = y_{10}^{(1)} + \mu \left[\frac{\partial y_1}{\partial t} t_{11}^{(1)} + \frac{\partial y_1}{\partial h_0} \alpha_{01}(t_{10}^{(1)}) + \frac{\partial y_1}{\partial t} \beta_{01}(t_{10}^{(1)}) \right] + \mu^2(\ldots)$$

Taking into account the fact that

$$y_{10}^{(1)} \frac{\partial y_1}{\partial h_0} + \psi_1 [x_1(h_0,\ t_{10}^{(1)} + \varphi_0)] \frac{\partial x_1}{\partial h_0} \equiv 1 \tag{8}$$

we obtain

$$y_1^{(1)} = y_{10}^{(1)} + \frac{\mu}{y_{10}^{(1)}} \int_{L_1^{(1)}} f_1^{(1)}(x,\ y)\, dx + \mu^2(\ldots) \tag{9}$$

where $L_1^{(1)}$ is a curve defined by (4) and passing through the points P_0 $(x_0,\ y_0)$ and $P_{10}^{(1)}$ $(x_1,\ y_{10}^{(1)})$.

Assuming that Formula (3) holds during the transformation of the line $S_0^{(1)}$ into $S_{k-1}^{(1)}$ we can show, that it also holds when $S_0^{(1)}$ goes into $S_k^{(1)}$ in the upper semiplane. Moreover, it holds when $S_0^{(1)}$ goes into $S_k^{(2)}$ (when the representative point passes through the straight line $y = 0$ on which the pieces of the function $f(x,\ y)$ are joined), and the argument which led to the latter statement applies fully to the transformation of the line $S_k^{(2)}$ (in the lower semiplane) into the initial line $S_0^{(1)}$ (in the upper semiplane).

Let us now assume that for $\mu = 0$, the system (1) has a family of periodic solutions $L(y_0)$ depending on the parameter y_0. Then the point transformation of the line $S_0^{(1)}$ into itself near the closed curve L, have the form

$$y_0^{(1)} = y_0 + \frac{\mu}{y_0} \int_L f(x,\ y)\, dx + \mu^2(\ldots) \equiv y_0 + \mu F(y_0) + \mu^2(\ldots) \tag{10}$$

where $L = L(y_0)$ is a closed integral curve passing through P_0 $(x_0,\ y_0)$.

We have two obvious theorems:

T h e o r e m 1. The condition

$$P_0\,(x_0,\; y_0^0 + \mu y_1)$$

is necessary and sufficient for the transformation (10) to have, at sufficiently small μ, a fixed point

$$F\,(y_0^0) = 0 \tag{11}$$

which tends to $P(x_0,\, y_0^0)$ as $\mu \to 0$.

Theorem 2. Let y_0^0 be a solution of (11). If

$$F'\,(y_0^0) \neq 0$$

then (10) has a fixed point

$$P_0\,(x_0,\; y_0^0 + \mu y_1)$$

which tends to $P(x_0,\, y_0^0)$ when $\mu \to 0$. This point is stable if $F'(y_0^0) < 0$ and unstable if $F'(y_0^0) > 0$.

The above conditions of existence and stability of periodic solutions of (1) are analogous to the corresponding conditions given in [1] for the systems which are almost Hamiltonian.

If the functions $\psi(x)$ and $f(x,\, y)$ are periodic in x and their period is 2π, then the phase space of (1) will be cylindrical with two similar lines $x = x_0$ and $x = x_0 + 2\pi$. Theorems 1 and 2 will then refer to the fixed point corresponding to the periodic solution enveloping the phase cylinder. The curve $L(y_0^0)$ will in this case be a closed integral curve of (1) with $\mu = 0$, it will pass through the point $(x_0,\, y_0)$ and envelope the phase cylinder.

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BIBLIOGRAPHY

1. Pontriagin, L.S., On dynamic systems which are almost Hamiltonian. ZhETF, Vol. 2, No. 3, 1934.

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